

Some Notes on the Baer-invariant of a Nilpotent Product of Groups*

Behrooz Mashayekhy

Department of Mathematics, Ferdowsi University of Mashhad,
P.O.Box 1159-91775, Mashhad, Iran
E-mail: mashaf@science2.um.ac.ir

and

Institute for Studies in Theoretical Physics and Mathematics,
Tehran, Iran

Abstract

W.Haebich (Bull. Austral. Math. Soc., 7, 1972, 279-296) presented a formula for the Schur multiplier of a regular product of groups. In this paper first, it is shown that the Baer-invariant of a nilpotent product of groups with respect to the variety of nilpotent groups has a homomorphic image and in finite case a subgroup of Haebich's type. Second a formula will be presented for the Baer-invariant

*This research was in part supported by a grant from IPM

of a nilpotent product of cyclic groups with respect to the variety of nilpotent groups.

A.M.S. Classification: 20F12,20F18,20K25

Key words and phrases: Baer-invariant, Nilpotent Product and Nilpotent Variety

1.INTRODUCTION

In 1907, I.Schur [], using representation method, showed that if G is the direct product of A and B , then the Schur multiplier of G has the following form :

$$M(G) = M(A \times B) \cong M(A) \times M(B) \times (A_{ab} \otimes B_{ab}).$$

Also, J.Wiegold [] in 1971 obtained the above result by some properties of covering groups:

$$M(A \times B) \cong M(A) \oplus M(B) \oplus \frac{[A, B]}{[A, B, A * B]}, \text{ where } \frac{[A, B]}{[A, B, A * B]} \cong A_{ab} \otimes B_{ab}.$$

In 1979, M.R.R.Moghaddam [] found a formula for the Baer-invariant (the generalization of the Schur multiplier with respect to the variety of groups.) of a direct product of two groups with respect to the variety of nilpotent groups of class at most c , \mathcal{N}_c , where $c + 1$ is a prime number or 4. This result generalized somehow the work of Schur and Wiegold.

In 1997, we [] presented an explicit formula for the Baer-invariant of a finite abelian group with respect to the variety of nilpotent groups for every $c \geq 1$. Since a finite abelian group is a direct product of some cyclic groups,

so this result is somehow a generalization of Schur-wiegold's result and also Moghaddam's result.

It is known that nilpotent product is a generalization of direct product. Therefore it is interesting to find a formula for the Schur multiplier or the Baer-invariant of a nilpotent product.

In 1972, W.Haebich [] (Theorem 2.14) presented a formula for the Schur multiplier of a regular product of a family of groups. We know that regular product is a generalization of nilpotent product, so this result is an interesting generalization of the Schur-Wiegold's result. Also, M.R.R.Moghaddam [] (Theorem 2.7) in 1979 gave a formula similar to Haebich's formula for the Schur multiplier of a nilpotent product. His approach was different from W.Haebich.

Finally, N.D.Gupta and M.R.R.Moghaddam [] (Theorem 2.15) in 1992 have calculated the Baer-invariant of the nilpotent dihedral group of class n

$$G_n = \langle x, y, |x^2, y^2, [x, y]^{2^{n-1}} \rangle$$

with respect to the variety of nilpotent groups, \mathcal{N}_c . It is routine to verify that G_n is isomorphic to the n th nilpotent product of two cyclic groups i.e $G_n = \mathbf{Z}_2 \overset{n}{*} \mathbf{Z}_2$.

2.NOTATION AND PRELIMINERIES

It is supposed that the reader is familiar with the notions of variety of groups, verbal and marginal subgroups, Schur multiplier and Baer-invariant. In particular if G is a group with a free presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1 ,$$

then the Baer-invariant of G with respect to the variety of nilpotent groups of class at most $c \geq 1$, \mathcal{N}_c , denoted by $\text{cal}\mathcal{N}_c M(G)$, is defined to be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, {}_c F]}$$

(See C.R.Leedham-Green and S.McKay [], from where our notation have been taken and H.Neumann [] for the notion of variety of groups.)

Clearly, if $c = 1$, then $\mathcal{N}_c = \mathcal{A}$ is the variety of abelian groups and

$$\mathcal{N}_c M(G) = M(G) \cong \frac{R \cap F'}{[R, F]}$$

is the *Schur multiplier* of G .

Definition 2.1

A variety \mathcal{V} is said to be a *Schur-Baer variety* if for any group G for which the marginal factor group, $G/V^*(G)$, is finite, then the verbal subgroup, $V(G)$, is also finite and $|V(G)|$ divides a power of $|G/V^*(G)|$. I.Schur in [11] proved that the variety of abelian groups, \mathcal{A} , is a Schur-Baer variety. Also R.Baer in [1] proved that the variety defined by some outer commutator words has the above property. The following theorem tells us a very important property of Schur-Baer varieties.

Theorem 2.2(C.R.Leedham-Green and S.McKay [6])

The following conditions on the variety \mathcal{V} are equivalent:

- (i) \mathcal{V} is a Schur-Baer variety.
- (ii) For every finite group G , its Baer-invariant, $\mathcal{V}M(G)$, is of order dividing a power of $|G|$.

Definition 2.3

Consider the following map

$$\begin{aligned}\varphi^* : \prod_{i \in I}^* A_i &\longrightarrow \prod_{i \in I}^\times A_i \\ a_1 a_2 \dots a_n &\longmapsto (a_1, a_2, \dots, a_n)\end{aligned}$$

which is a natural map from the free product of $\{A_i\}_{i \in I}$ onto the direct product of $\{A_i\}_{i \in I}$. Clearly its kernel is the normal closure of

$$< [A_i, A_j] \mid i, j \in I, i \neq j >$$

in the free product $\prod_{i \in I}^* A_i$ and denoted by $[A_i]^*$ and called *the cartesian subgroup* of the free product.

For the properties of cartesian subgroup see H. Neumann [].

Definition 2.4

Let \mathcal{V} be a variety of groups defined by a set of laws V then *the verbal product* of a family of groups $\{A_i\}_{i \in I}$ associated with the variety \mathcal{V} is defined to be

$$\mathcal{V} \prod_{i \in I} A_i = \frac{\prod_{i \in I}^* A_i}{V(A) \cap [A_i]^*}.$$

The verbal product is also known as *varietal product* or simply \mathcal{V} -product. If \mathcal{V} is the variety of all groups, then the corresponding verbal product is the *free product*; if $\mathcal{V} = \mathcal{A}$ is the variety of all abelian groups, then the verbal product is the *direct product*. We denote $\mathcal{V} \prod_i A_i$ for the \mathcal{V} -product of the family of groups A_i , $i \in I$.

The \mathcal{V} -product of the A_i 's is 'between' the free product and the direct product in the sense that the natural epimorphism of the free product onto

the direct product can be factored through the \mathcal{V} -product, i.e

$$\prod_i^* A_i \xrightarrow{\nu} \mathcal{V} \prod_i A_i \xrightarrow{\varphi(\nu)} \prod_i^\times A_i .$$

The kernel of the natural epimorphism $\varphi(\nu)$ of the \mathcal{V} -product onto the direct product of the A_i 's is the cartesian subgroup of $\mathcal{V} \prod_i A_i$, that is the normal closure of $\langle [A_i, A_j] \mid i \neq j, i, j \in I \rangle$, in the \mathcal{V} -product, which is denoted by $[A_i]^V$.

Lemma 2.5

With the above notation, we have

- (i) $\text{Ker} \varphi(\nu) = [A_i]^V = \nu([A_i]^*)$ where $\varphi^* = \varphi(\nu) \circ \nu$ as given in 2.7 .
- (ii) If $a \neq 1$ is an element of $\mathcal{V} \prod_i A_i$, then

$$a = a_{i_1} \dots a_{i_m} c , \quad 1 \neq a_{i_j} \in A_{i_j} , \quad i_1 < i_2 < \dots < i_m , \quad c \in [A_i]^V ,$$

and the a_{i_j} and c are uniquely determined by a and chosen order of I .

Proof. See [] .

Definition 2.6

Let $\{A_i\}_{i \in I}$ be a family of groups and \mathcal{N}_c be the variety of nilpotent groups of class at most $c \geq 1$. Then the \mathcal{N}_c -product, $\mathcal{N}_c \prod_i A_i$, is called the *cth nilpotent product* of A_i 's. In particular, if A and B are two groups, then the *cth nilpotent product* of A and B , denoted by $A \overset{c}{*} B$, is as follows:

$$A \overset{c}{*} B = \frac{A * B}{[A, B, \overset{c-1}{[A * B]}]} .$$

Theorem 2.7 (M.R.R.Moghaddam [9])

Let A and B be two groups and $A \overset{n}{*} B$ be the *nth nilpotent product* of A and B . Then for $n \geq 1$

$$M(A \overset{n}{*} B) \cong M(A) \oplus M(B) \oplus \frac{[A, B, \overset{(n-1)}{[A * B]}]}{[A, B, \overset{n}{[A * B]}]} .$$

Definition 2.8

Let $\{A_i\}_{i \in I}$ be a family of subgroups of an arbitrary group G . We say that G is a *regular product* of its subgroups A_i 's, $i \in I$, where I is an ordered set if the following two conditions hold:

- (i) $G = \langle A_i \mid i \in I \rangle$,
- (ii) $A_i \cap \hat{A}_i = 1$ for all $i \in I$, where $\hat{A}_i = \prod_{j \neq i, j \in I} A_j^G$

The subgroups A_i 's, $i \in I$ will be referred to as the *regular factors* of the group G . The direct product, the free product, and the verbal product of an arbitrary set of groups are examples of regular products.

Lemma 2.9

Let A be a subgroup of a group G , and N be a normal subgroup of G and $\{M_i \mid i \in I\}$ be a family of normal subgroups of G . Then

$$[A \prod_i M_i, N] = [A, N] \prod_i [M_i, N].$$

The following theorem gives a characterization of regular product .

Theorem 2.10 (O.N.Golovin [2])

Suppose that a group G is generated by a family $\{A_i \mid i \in I\}$ of its subgroups, where I is an ordered set. Then G is a regular product of the A_i 's if and only if every element of G can be written uniquely as a product

$$a_{\lambda_1} a_{\lambda_2} \dots a_{\lambda_m} u ,$$

where

$$1 \neq a_{\lambda_i} \in A_{\lambda_i}, \lambda_i < \lambda_2 < \dots < \lambda_m$$

and

$$u \in [A_i^G] = \langle [A_\lambda^G, A_\mu^G] \mid \lambda \neq \mu, \lambda, \mu \in I \rangle .$$

Proof. See [2,5] .

Definition 2.11

Let G be a regular product of A_i 's, $i \in I$. A homomorphism $f : G \rightarrow H$ of G to a group H is called a *regular homomorphism* if

$$\text{Ker } f \subseteq [A_i^G] .$$

The groups $\text{Im } f$ and $G/\text{Ker } f$ are called a *regular homomorphic image* and a *regular quotient group*, respectively.

Theorem 2.12

If G is a regular product of the subgroups A_i 's, $i \in I$, and $f : G \rightarrow H$ is a regular homomorphism, then f restricted to A_i is an *isomorphism* for each $i \in I$ and $f(G)$ is a regular product of $f(A_i)$.

Proof. See [10] .

Theorem 2.13

Let G be generated by the A_i 's, $i \in I$, and let

$$\psi : \prod_{i \in I}^* A_i \longrightarrow G$$

be the homomorphism induced by the identity map on each A_i . Then G is a regular product of the A_i 's if and only if ψ is regular.

Proof. See [].

Theorem 2.14 (W.Haebich [])

Let G be a regular product of a family of its subgroups $\{A_i | i \in I\}$ and let H be the kernel of the natural homomorphism

$$A = \prod_i^* A_i \longrightarrow G ,$$

induced by the identity map on each $A_i, i \in I$. Then

$$M(G) \cong \left(\prod_{i \in I} M(A_i) \right) \oplus \frac{H}{[H, A]}.$$

Theorem 2.15(N.D.Gupta and M.R.R.Moghaddam [])

Let G_n be the nilpotent dihedral group of class n , i.e $G_n = \mathbf{Z}_2 \overset{n}{*} \mathbf{Z}_2$, then

(i) If $c < n$, then $\mathcal{N}_c M(G_n)$ is the cyclic group of order 2^c . (ii) If $c \geq n$, then $\mathcal{N}_c M(G_n)$ is an abelian extension of a cyclic group of order 2^{n-1} by an elementary abelian 2-group of rank $r(c+1)$, the number of basic commutators of wieght $c+1$ on two letters. In particular, for $c \geq n$,

$$\mathcal{N}_c M(G_n) \cong \mathbf{Z}_{2^n} \oplus \mathbf{Z}_2 \oplus \dots \oplus \mathbf{Z}_2 \text{ (} r(c+1) - 1 \text{ copies of } \mathbf{Z}_2 \text{)}.$$

Proof. See [] Theorem .

Lemma 2.16

If H is a subgroup of a finite abelian group G , then G has a subgroup isomorphic to G/H .

3. A SUBGROUP OF HAEBICH'S TYPE

Let $\{A_i | i \in I\}$ be a family of groups (we can consider I as an ordered set) and for each $i \in I$, F_i denote a fixed free group such that the following exact sequence be a free presentation for A_i

$$1 \longrightarrow R_i \longrightarrow F_i \xrightarrow{\nu_i} A_i \longrightarrow 1.$$

We denote by ν the natural homomorphism from the free product $F = \prod_{i \in I}^* F_i$ onto $A = \prod_{i \in I}^* A_i$ induced by the ν_i 's. Also, the group G_n will be assumed to be the n th nilpotent product of the A_i 's, $i \in I$. If ψ_n is the

natural homomorphism from A onto G_n induced by the identity map on each A_i ;

$$F = \prod_{i \in I}^* F_i \xrightarrow{\nu} A = \prod_{i \in I}^* A_i \xrightarrow{\psi_n} G_n = \prod_{i \in I}^n A_i \longrightarrow 1 ,$$

then we denote by H_n be the kernel of ψ_n and L_n the inverse image of H_n in F under ν ; $H_n = \ker \psi_n$ & $L_n = \nu^{-1}(H_n)$.

Now we have the following lemma.

Lemma 3.1

By the above notation and assumption, we have

- (i) $H_n = \nu(K_n)$, where $K_n = \gamma_{n+1}(F) \cap [F_i]^*$
- (ii) $G_n \cong F/L_n$, where $L_n = \ker(\psi_n \circ \nu) = (\prod_{i \in I} R_i^F)K_n$

Proof.

- (i) By definition of n th nilpotent product, we have $H_n = \gamma_{n+1}(A) \cap [A_i]^*$, where $[A_i]^*$ is the cartesian subgroup of A , so clearly $\nu(K_n) = H_n$.
- (ii) Clearly $\ker \nu = \prod_{i \in I} R_i^F$, and composition $\psi_n \circ \nu : F \longrightarrow G_n$ is surjective homomorphism. Since $H_n = \ker \psi_n = \nu(K_n)$, it follows that

$$L_n = \ker(\psi_n \circ \nu) = (\ker \nu)K_n = (\prod_{i \in I} R_i^F)K_n .$$

Hence $G_n \cong F/L_n$. \square

Lemma 3.2

If $D_1 = \prod_{i \neq j} [R_i, F_j]^F$, then $\prod_{i \in I} R_i^F = (\prod_{i \in I} R_i)D_1$.

Proof.

See [] Lemma .

Notation 3.3

- We define (i) $D_c = \prod_{\exists j, \mu_j \neq i} [R_i, F_{\mu_1}, \dots, F_{\mu_c}]^F$,
(ii) $E_c = D_1 \cap \gamma_{c+1}(F)$.

Clearly $D_c \subseteq E_c$ for every $c \geq 1$.

Now we prove the following important lemma.

Lemma 3.4

By the above notation and assumption, we have

- (i) $[L_n, {}_c F] = (\prod_{i \in I} [R_i, {}_c F_i]) D_c [K_n, {}_c F]$.
(ii) If $n \leq c$, then $L_n \cap \gamma_{c+1}(F) = (\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i))) E_c K_c$.
(iii) If $n > c$, then $L_n \cap \gamma_{c+1}(F) = (\prod_{i \in I} (R_i \cap \gamma_{c+1}(F_i))) E_c K_n$.

Proof.

- (i) By definition of $C_{(n)}$, we have

$$\begin{aligned} [C_{(n)}, {}_c B] &= [(\prod_{i \in I} C_i^B) K_n, {}_c B] = [(\prod_{i \in I} C_i^B), {}_c B] [K_n, {}_c B] \quad (\text{by Lemma 2.18}) \\ &= (\prod_{i \in I} [C_i, {}_c B]^B) [K_n, {}_c B] \quad (\text{by Lemma 2.18}) \\ &\subseteq (\prod_{i \in I} [C_i, {}_c B_i]) D_c [K_n, {}_c B] \quad (\text{by definition of } D_c) \end{aligned}$$

The reverse inclusion is obviously true, so the result holds.

- (ii) Let $n \leq c$, and $g \in C_{(n)} \cap \gamma_{c+1}(B)$, by Lemmas 3.1 and 3.2 we have

$$g = c_{\lambda_1} \dots c_{\lambda_t} dk ,$$

where $c_{\lambda_i} \in C_{\lambda_i}$, $d \in D_1$, $k \in K_n$. Now consider the following homomorphism

$$B = \prod_{i \in I} {}^* B_i \xrightarrow{\text{nat}} \prod_{i \in I} {}^\times B_i .$$

Since $g \in \gamma_{c+1}(B)$, the image of g under the above natural homomorphism is

$$(c_{\lambda_1}, \dots, c_{\lambda_t}) \in \gamma_{c+1}(\prod^\times B_i) = \prod^\times \gamma_{c+1}(B_i) .$$

Therefore $c_{\lambda_i} \in \gamma_{c+1}(B_{\lambda_i}) \cap C_{\lambda_i}$ and so $dk \in \gamma_{c+1}(B) \cap [B_i]^* = K_c$.

The reverse inclusion can be seen easily, and so the result holds.

(iii) It is clear. \square

References

- [1] R.Baer: "Endlichkeitskriterien für Kommutatorgruppen." Math. Ann., 124,161-177 (1952).
- [2] O.N.Golovin: " On the Isomorphisms of Nilpotent Decompositions of Groups." Mat. Sbornik N.S. 28(70),445-52 (1951) Amer. Math. Soc. Translations,Ser. 2,2,133-140 (1956).
- [3] W.Haebich: "The Multiplicator of a Regular Product of Groups." Bull. Austral. Math. Soc.,7,279-296,(1972).
- [4] T.W.Hungerford: "Algebra" Springer-Verlag,G.T.M. 73 (1980).
- [5] G.Karpilovsky: "The Schur Multiplier." London Math. Soc. Monographs,New Series no. 2,(1987).
- [6] C.R.Leedham-Green and S.McKay: "Baer-invariant,Isologism, Varietal Laws and Homology." Acta Math.,137,99-150 (1976).

- [7] B.Mashayekhy and M.R.R.Moghaddam:“Higher Schur Multipliator of a Finite Abelian Group”,Algebra Colloquium, 4:3, 317-322, 1997.
- [8] M.R.R.Moghaddam:“The Baer-invariant of a Direct Product.” Arch. der. Math.,33,504-511,(1979).
- [9] M.R.R.Moghaddam:“The Schur Multipliator of Nilpotent Products.” Bull. Iranian math. Soc. ,vol. 6,no. 2,79-89,(1979).
- [10] H.Neumann:“Varieties of Groups.” Berlin,Springer-Verlag,(1967).
- [11] I.Schur:“Über die Darstellung der Endlichen Gruppen durch Gebrochene Lineare Substitutionen.” J. Reine Angew. Math.,127,20-50,(1904).
- [12] I.Schur:“Untersuchungen über die Darstellung der Endlichen Gruppen durch Gebrochene Lineare Substitutionen.” J. Reine Angew. Math.,132, 85-137,(1907).
- [13] R.F.Turner-Smith:“Marginal Subgroup Propeties for Outer Commutator Words.” Proc. London Math. Soc.,14,321-324,(1964).
- [14] J.Wiegold:“The Multiplicator of a Direct Product.” Quart. J. Math. Oxford,(2),22,103-105,(1971).